MMAT5030 Notes 7

1. The Method of Separation of Variables

In the last lecture we studied the initial-boundary problem for the heat equation under the Dirichlet condition (the temperature is fixed to zero at both ends of the rod)

$$\begin{cases} u_t = \kappa u_{xx} & \text{in } [0,l]\times(0,\infty)\ , \\ u(x,0)=f(x) & \text{in } [0,l], \\ u(x,t)=0 & \text{at } x=0,\ l \text{ and } t>0, \end{cases} \tag{1}$$

as well as under the Neumann condition (the rod is insulated at both ends).

$$\begin{cases} u_t = \kappa u_{xx} & \text{in } [0,l]\times(0,\infty),\ \kappa>0, \\ u(x,0)=f(x) & \text{in } [0,l], \\ u_x(x,t)=0 & \text{at } x=0,l \text{ and } t>0. \end{cases} \tag{2}$$

It turns out the first problem can be solved in terms of a sine series while the second one in terms of a cosine series. However, when it comes to more general boundary conditions, a straightforward appeal to Fourier series may not work. To illustrate this point, let us consider the problem

$$\begin{cases} u_t = u_{xx} & \text{in } [0,\pi]\times(0,\infty)\ , \\ u(x,0)=f(x) & \text{in } [0,\pi], \\ u_x(0,t)=0, \quad u(\pi,t)=0, & t>0, \end{cases} \tag{3}$$

We have simplified the problem by setting $\kappa = 1$ and $l = \pi$. It does no harm in the mathematical point of view, since the general case can be reduced to this normalized case by rescaling. However, even in a simplified form, it is not clear at all how to represent the solution as a cosine, a sine or a Fourier series. We will use the method of separation of variables to solve the problem.

The decisive step in this method is to find all special solutions of the form $X(x)T(t)$. Indeed to satisfy the equation $u_t = u_{xx}$, we have

$$T'(t)X(x) = T(t)X''(x).$$

Dividing both sides of this equation by TX, we have

$$\frac{T'}{T} = \frac{X''}{X}.$$

As the left hand side of this equation is a function of t while its right hand side depends only on x, we must have $T' = -\lambda T$ and $X'' = -\lambda X$ for some constant λ. Taking account into the boundary conditions, the function X must satisfy the "eigenvalue problem"

$$\begin{cases} X'' + \lambda X = 0, \\ X'(0) = 0, \quad X(\pi) = 0 \ . \end{cases} \tag{4}$$

We look for those λ so that this problem admits non-zero (or nontrivial) solutions. We note that this is a homogeneous differential equation with homogeneous boundary conditions. Therefore, if X is solution, every scalar multiple of X is a also a solution. We divide into three cases according to the sign of λ.

Case (i) $\lambda > 0$. In this case the general solution of the equation is given by

$$X(x) = C\cos\mu x + D\sin\mu x, \quad \mu = \sqrt{\lambda} \ .$$

We have $X'(x) = -\mu C\sin\mu x + \mu D\cos\mu x$. So $0 = X'(0) = \mu D$ implies $D = 0$. Next $0 = X(\pi) = C\cos\mu\pi$ implies $\mu\pi = (n+1/2)\pi$, that is, $\mu = n + 1/2, n \geq 0$.

Case (ii) $\lambda = 0$. In this case $X(x) = C + Dx$. $X'(0) = 0$ implies $D = 0$ and $X(\pi) = 0$ implies $C = 0$. There is no non-trivial solution in this case.

Case (iii) $\lambda < 0$. The general solution $X(x) = Ce^{\mu x} + De^{-\mu x}$, $\mu = \sqrt{-\lambda}$. We have $X'(x) = C\mu e^{\mu x} - D\mu e^{-\mu x}$. The boundary conditions are

$$C - D = 0 \ , \quad Ce^{\mu\pi} + De^{-\mu\pi} = 0 \ .$$

The first equation shows that $C = D$. By plugging this into the second equation, $(e^{\mu\pi} + e^{-\mu\pi})C = 0$, which implies $C = D = 0$. Again there is no non-trivial solution.

In conclusion, this eigenvalue problem admits non-trivial solutions if and only if $\lambda_n = (n+1/2)^2, n \geq 0$, and the corresponding eigenfunction is a non-zero multiple of

$$X_n(x) = \cos(n+1/2)x.$$

From $T'/T = -\lambda_n$, we get $T(t) = ce^{-(n+1/2)^2 t}$, $c \neq 0$. It follows that the heat equation together with $u_x(0,t) = 0$ and $u(\pi,t) = 0$ admits infinitely many solutions given by

$$e^{-(n+1/2)^2 t}\cos(n+1/2)x \ , \quad n \geq 0 \ .$$

Bearing in mind that every non-zero multiple of these solutions is again a solution. We let the formal solution of this new initial-boundary problem (3) be

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(n+1/2)^2 t}\cos(n+1/2)x \ , \quad A_n \in \mathbb{R}.$$

In order that this solution satisfies the initial value, it is necessary to assume the initial function admits the following expansion

$$f(x) = \sum_{n=0}^{\infty} A_n \cos(n + 1/2)x \ .$$

At this point the method of separation of variables has completed its job. The rest is now the job of hard analysis.

Actually, there is a tricky way to solve this problem by extending the solution as a 4π-periodic function, see exercise. In the following we use separation of variables to study more complicated boundary conditions. It shows that Fourier series cannot be applied any more.

Consider the (normalized) heat equation subject to Robin boundary condition

$$\begin{aligned} &u_x - au = 0 \qquad \text{at } \ x = 0, \\ &u_x + bu = 0 \qquad \text{at } \ x = \pi, \quad a, b > 0. \end{aligned} \tag{5}$$

Again we first look for separated solutions $X(x)T(t)$. The same as in the previous example, X must satisfy the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0, \\ X' - aX = 0 \quad \text{at } 0, \qquad X' + bX = 0 \quad \text{at } \pi. \end{cases} \tag{6}$$

In the following we solve this eigenvalue problem. To simplify the situation, let's assume $\sqrt{ab} \neq n + 1/2$ for any $n \geq 0$. All the remaining cases are discussed in W Strass' book "Partial Differential Equations".

Case (i) $\lambda > 0$. The general solution is again

$$X(x) = C \cos \mu x + D \sin \mu x, \quad \mu = \sqrt{\lambda} > 0,$$

where C and D are arbitrary constants. To satisfy the boundary conditions we need to find nontrivial C and D so that

$$\begin{cases} -aC + \mu D = 0 \\ (-\mu \sin \mu\pi + b \cos \mu\pi)C + (\mu \cos \mu\pi + b \sin \mu\pi)D = 0. \end{cases}$$

To have a non-zero pair of solution (C, D) it is required

$$\det \begin{pmatrix} -a & \mu \\ -\mu \sin \mu\pi + b \cos \mu\pi & \mu \cos \mu\pi + b \sin \mu\pi \end{pmatrix} = 0,$$

i.e.,

$$-a(\mu \cos \mu\pi + b \sin \mu\pi) + \mu(\mu \sin \mu\pi - b \cos \mu\pi) = 0 \ ,$$

or

$$-(a+b)\mu\cos\mu\pi + (\mu^2 - ab)\sin\mu\pi = 0 \ .$$

Observe that $\cos\mu\pi = 0$ iff $\mu = n + 1/2$. When this happens, the equation above implies that $\mu^2 = (n+1/2)^2 = ab$ which is in conflict with our assumption $\sqrt{ab} \neq n+1/2$ for any $n \geq 0$. Hence $\cos\mu\pi \neq 0$. We can divide the above equation by $\cos\mu\pi$ to get

$$\tan\mu\pi = \frac{(a+b)\mu}{\mu^2 - ab} \tag{7}$$

We conclude that (6) has a non-zero solution if and only if μ satisfies (7). By plotting graphs (examining the intersections of the curve $y = \tan\pi\mu$ and the curve $y = (a+b)\mu/(\mu^2 - ab)$ regarding μ as the independent variable), we see that there are infinitely many solutions where the corresponding μ_n satisfy $n-1 < \mu_n < n$, or $(n-1)^2 < \lambda_n < n^2, n \geq 1$. Corresponding to λ_n, the solution is given by a non-zero multiple of

$$\cos\mu_n x + \frac{a}{\mu_n}\sin\mu_n x \ ,$$

after using the first boundary condition $-aC + \mu D = 0$, that is, $D = a/\mu$ when $C = 1$.

Case (ii) $\lambda = 0$. The general solution is given by

$$X(x) = C + Dx,$$

where the constants C and D satisfy

$$D - aC = 0, \quad (1+\pi b)D + bC = 0.$$

The determinant is given by

$$\det\begin{pmatrix} 1 & -a \\ 1+b\pi & b \end{pmatrix},$$

that is, $(a+b) + ab\pi$ which is not equal to 0. Therefore, this linear system can only admit zero solution.

Case (iii) $\lambda < 0$. The general solution is

$$X(x) = Ce^{\mu x} + De^{-\mu x} \ , \mu = \sqrt{\lambda} \ .$$

To satisfy the boundary conditions, C and D should satisfy

$$C(\mu - a) - D(\mu + a) = 0, \quad C(\mu+b)e^{2\mu\pi} - D(\mu - b) = 0 \ .$$

We have

$$\det\begin{pmatrix} \mu - a & \mu + a \\ e^{2\mu\pi}(\mu+b) & \mu - b \end{pmatrix} = 0,$$

that is,

$$\frac{\mu^2 - (a+b)\mu + ab}{\mu^2 + (a+b)\mu + ab} = e^{2\mu\pi} \ .$$

Noting that the right hand side is always greater than 1 and the left hand side cannot exceed 1, we see that this linear system has no non-zero solution.

We conclude that all separated solutions are given by non-zero multiples of

$$X_n(x) = \cos \mu_n x + \frac{a}{\mu_n} \sin \mu_n x \ , n \geq 1 \ .$$

The following orthogonality result is a positive evidence showing a "Fourier series theory" may be developed for these functions.

Proposition 7.1. *For $n, m, n \neq m$,*

$$\int_0^\pi X_n(x) X_m(x) dx = 0 \ .$$

Proof. It is tedious to check the orthogonality condition directly. Instead we use an inspiring argument. Since both X_m and X_n satisfy (6) for $\lambda = \mu_n^2$ and μ_m^2 respectively, we have

$$\begin{aligned}
-\mu_m^2 \int_0^\pi X_m(x) X_n(x) dx &= \int_0^\pi X_m''(x) X_k(x) dx \\
&= X_m'(x) X_n(x) \Big|_0^\pi - \int_0^\pi X_m'(x) X_n'(X) dx \\
&= X_m'(x) X_n(x) \Big|_0^\pi - X_m(x) X_n'(x) \Big|_0^\pi + \int_0^\pi X_m(x) X_n''(x) dx \\
&= -\mu_n^2 \int_0^\pi X_m(x) X_n(x) dx \ ,
\end{aligned}$$

where we have used the boundary conditions in (6) to cancel the two boundary terms. As $\mu_n \neq \mu_m$, it forces

$$\int_0^\pi X_m(x) X_n(x) dx = 0 \ .$$

□

By multiplying with a suitable constant c_n, the functions $\varphi_n = c_n X_n$ satisfy

$$\|\varphi_n\| \equiv \sqrt{\int_0^\pi \varphi_n^2(x) dx} = 1,$$

for all n, hence $\{\varphi_n\}$ forms an orthonormal set in $R[0, \pi]$.

Naturally one hopes that the solution of the initial-boundary problem for the heat equation under the Robin condition is

$$u(x,t) = \sum_{n=1}^\infty A_n e^{-\lambda_n t} \varphi_n(x).$$

In order $u(x,t)$ satisfies the initial condition we require A_n be determined from the expansion

$$f(x) = \sum_{n=1}^{\infty} A_n \varphi_n(x),$$

or equivalently, using Proposition 7.1,

$$A_n = \int_0^{\pi} f(x)\varphi_n(x)dx.$$

This approach can be fully justified if we can establish that $\{\varphi_n\}$ is a complete orthonormal set in $R[0,\pi]$ as well as that those nice convergence results concerning Fourier series hold in the present context. All this could be done by more sophisticated methods, we will not elaborate on this point. We will be satisfied as long as the formal solution is found.

2. The Dirichlet Problem for the Laplace Equation

The Laplace equation

$$\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \ ,$$

is perhaps the most important partial differential equation. It arises from various context. For instance, the temperature distribution in a plane set satisfies the two dimensional heat equation

$$u_t = \kappa \Delta u$$

for some $\kappa > 0$. As time tends to ∞, the temperature distribution would tend to an equilibrium state which is independent of time. So the equilibrium state $u(x)$ satisfies the Laplace equation $\Delta u = 0$ in this plane set. Be cautious it is not always equal to a constant because the non-triviality of the boundary data. In the one dimensional case, the temperature distribution on the rod eventually satisfies $u_{xx} = 0$, which can be solved readily to get $u(x) = a + bx$, that is, it is linear function. However, in higher dimensions there are many many functions satisfying the Laplace equation. These functions are called harmonic functions. It is interesting to see that the method of separation of variables can be applied to the Dirichlet problem for the two dimensional Laplace equation on the disk

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = \varphi & \text{on } C, \end{cases} \tag{8}$$

where $D = \{(x,y) : x^2 + y^2 < 1\}$ and C is the unit circle. The trick of solving (8) is to write things in the polar coordinates. We let $u(x,y) = v(r,\theta)$ where the variables are related by

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x} \ .$$

By differentiating the relation $u(x,y) = v(r,\theta)$, we use the chain rule in advanced calculus to get

$$u_x = v_r \frac{\partial r}{\partial x} + v_\theta \frac{\partial \theta}{\partial x} \ ,$$

$$u_{xx} = \left(v_{rr} \frac{\partial r}{\partial x} + v_{r\theta} \frac{\partial \theta}{\partial x} \right) \frac{\partial r}{\partial x} + v_r \frac{\partial^2 r}{\partial x^2} + \left(v_{\theta\theta} \frac{\partial \theta}{\partial x} + v_{\theta r} \frac{\partial r}{\partial x} \right) \frac{\partial \theta}{\partial x} + v_\theta \frac{\partial^2 \theta}{\partial x^2} \ .$$

Adding up with a similar formula for u_{yy}, we have

$$\begin{aligned} \Delta u &= v_{rr} \left[\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right] + 2 v_{r\theta} \left(\frac{\partial \theta}{\partial x} \frac{\partial r}{\partial x} + \frac{\partial \theta}{\partial y} \frac{\partial r}{\partial y} \right) \\ &\quad + \left[\left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \theta}{\partial y} \right)^2 \right] v_{\theta\theta} + v_r \Delta r + v_\theta \Delta \theta \ . \end{aligned}$$

Using

$$\theta_x = \frac{-y}{r}, \quad \theta_y = \frac{x}{r}, \quad r_x = \frac{x}{r}, \quad r_y = \frac{y}{r} \ ,$$

and

$$\theta_{xx} = \frac{2xy}{r^4}, \quad \theta_{yy} = \frac{-2yx}{r^4}, \quad r_{xx} = \frac{y^2}{r^3}, \quad r_{yy} = \frac{x^2}{r^3} \ ,$$

we obtain

$$\Delta u = \frac{\partial^2 v}{\partial r^2} +, \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \ .$$

Traditionally people do not make a difference between u and v in notation. In the following we use u to represent the same function no matter the independent variables are x, y or r, θ. The boundary value problem (8) is now transformed to

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial u}{\partial r} + \dfrac{1}{r^2} \dfrac{\partial^2 u}{\partial \theta^2} = 0 & \text{in } R, \\ u(1,\theta) = \varphi(\theta) & \theta \in [-\pi, \pi], \end{cases} \tag{9}$$

where $R = \{(r,\theta), 0 \le r < 1, \theta \in [-\pi,\pi]\}$ is a rectangle and φ is 2π-periodic. We seek separated solutions of the form $R(r)\Theta(\theta)$. Plugging it into (9), we obtain

$$\Theta \left(R'' + \frac{1}{r} R' \right) + \frac{R}{r^2} \Theta'' = 0.$$

In other words,

$$\frac{r^2}{R} \left(R'' + \frac{1}{r} R' \right) = -\frac{\Theta''}{\Theta}.$$

As in the cases before we must have

$$R'' + \frac{1}{r} R' = \frac{-\lambda}{r^2} R,$$

and

$$\Theta'' - \lambda\Theta = 0,$$

for some constant λ. As Θ is a 2π-periodic function, it forces $\lambda = -n^2$, $n = 0, 1, 2, \cdots$. For $n \geq 1$,

$$\Theta(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad a_n, b_n \text{ constants.}$$

It is a constant when $n = 0$. Correspondingly we have

$$r^2 R_n'' + rR_n' - n^2 R_n = 0.$$

This ordinary differential equation is readily solved to get

$$R_n = c_1 r^n + c_2 r^{-n}, \qquad c_1, c_2 \text{ constants,}$$

(see exercise). Since $R_n(r)$ is at least continuous inside D and in particular at the origin, c_2 must vanish. When $n = 0$, R_0 is a constant. We conclude that any special solution of the Laplace equation of the form $R(r)\Theta(\theta)$ is given by a scalar multiple of

$$r^n(a_n \cos n\theta + b_n \sin n\theta), \quad n \geq 0.$$

In case the boundary data φ has the Fourier expansion

$$\varphi(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \ , \tag{10}$$

our solution is given by

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta). \tag{11}$$

We have

Theorem 7.2 *For every piecewise smooth, continuous 2π-periodic φ whose Fourier series is given by (10), the function defined in (11) is continuous in $D \cup C$, infinitely differentiable in D, and solves (8)/(9).*

You may skip the following proof which is similar to that of Theorem 5 in Notes 6.

Proof. In terms of polar coordinates, it suffices to show the function is continuous in $R \cup \partial R$ (∂R is the boundary of R) and infinitely differentiable inside R. To establish continuity, we observe that the Fourier series of φ' is given by

$$\varphi'(\theta) \sim \sum_{n=1}^{\infty} (nb_n \cos n\theta - na_n \sin n\theta) \ .$$

By Parseval's identity or Bessel's inequality,

$$\sum_{n=1}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq C \ ,$$

for some constant C depending on $\|\varphi'\|$. It follows that

$$\begin{aligned}\sum_{n=1}^{\infty}(|a_n|+|b_n|) &= \sum_{n=1}^{\infty}(n|a_n|+n|b_n|)n^{-1} \\ &\leq \frac{1}{2}\sum_{n=1}(n^2|a_n|^2+n^2|b_n|^2)+\sum_{n=1}^{\infty}\frac{1}{n^2} \\ &< \infty\ .\end{aligned}$$

Using

$$|r^n(a_n\cos n\theta+b_n\sin n)\theta| \leq |a_n|+|b_n|$$

and the above estimate, we can apply the M-test to the domain $R\cup\partial R$ to conclude that the series defined in the right hand side of (11) converges uniformly to the function u in $R\cup\partial R$. By Continuous Theorem (Notes 6) we know that u is continuous in $R\cup\partial R$. Also by pointwise convergence and (10) $u(1,\theta)=\varphi(\theta)$, that is, u satisfies the boundary condition.

It remains to show that (11) is infinitely differentiable inside D. In this connection we would like to apply the Differentiation Theorem (Notes 6). Hence it suffices to verify the series obtained by differentiating the right hand side of (11) finitely many times are uniformly convergent in a smaller disk

$$D(r_0)=\{(r,\theta):\ r\in[0,r_0],\ \theta\in[-\pi,\pi]\ \}\ .$$

By differentiating the right hand side of (11) k-many times in r we get

$$\sum_{n=k}^{\infty}C(n,k)r^{n-k}\left(a_n\cos nx+b_n\sin nx\right)\ ,\quad C(n,k)=n(n-1)\cdots(n-k+1)\ .$$

Next we differentiate it in θ finitely many times. Depending on whether the number is even or odd, we have

$$\sum_{n=k}^{\infty}C(n,k)(-1)^m n^{2m}r^{n-k}\left(a_n\cos nx+b_n\sin nx\right)\ ,$$

and

$$\sum_{n=k}^{\infty}C(n,k)(-1)^m n^{2m+1}r^{n-k}\left(a_n\sin nx+b_n\cos nx\right)\ .$$

Noting that $C(n,k)n^{2m+1}\leq n^{k+2m}$ and

$$\left(n^{k+2m}r_0^{n-k}\right)^{1/n}\to r_0<1,\qquad \text{as } n\to\infty\ ,$$

taking $\varepsilon=(1-r_0)/2$, there is some n_0 such that

$$\left|(n^{k+2m}r_0^{n-k})^{1/n}-r_0\right|<\frac{1-r_0}{2}\ ,\quad \forall n\geq n_0\ .$$

It follows that

$$0 \leq n^{k+2m} r_0^{n-k} < \left(\frac{1+r_0}{2}\right)^n \to 0 \ ,$$

as $n \to \infty$. We can find some n_1 such that $n^{k+2m} r_0^{n-k} \leq 1$ for $n \geq n_1$.

$$C(n,k) n^{2m+1} r^{n-k} \left(|a_n| + |b_n|\right) \leq |a_n| + |b_n|, \quad \forall n \geq n_1 \ .$$

By the M-test is convergent in $D(r_0)$. The same treatment applies to the second series as well. By the M-test we conclude that both series converge uniformly in $D(r_0)$. □

It is amazing that the solution given by (11) can be written in closed form. In fact, recalling that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(\alpha) \cos n\alpha d\alpha \ , \quad n \geq 0 \ ,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(\alpha) \sin n\alpha d\alpha, \quad n \geq 1 \ ,$$

we have

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\alpha) d\alpha + \sum_{n=1}^{\infty} \frac{r^n}{\pi} \int_{-\pi}^{\pi} \phi(\alpha)(\cos n\alpha \cos n\theta + \sin n\alpha \sin n\theta) d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} \phi(\alpha)[1 + 2\sum_{n=1}^{\infty} r^n \cos n(\theta - \alpha)] d\alpha. \end{aligned}$$

Using the formula (see exercise)

$$1 + 2\sum_{n=1}^{\infty} r^n \cos nx = \frac{1-r^2}{1 - 2r\cos x + r^2},$$

we arrive at the Poisson's formula

$$u(r,\theta) = \frac{1-r^2}{2\pi} \int_{-\pi}^{\pi} \frac{\phi(\alpha) d\alpha}{1 - 2r\cos(\theta - \alpha) + r^2} \ . \tag{12}$$

Using Poisson's formula we can prove the following sharpening version of Theorem 7.2.

Theorem 7.3. *For every continuous, 2π-periodic function φ, (12) defines an infinitely differentiable function in D which solves the equation in (8). Furthermore, $u(r,\theta)$ tends to $\varphi(\theta)$ as $r \to 1$.*

Comparing with Theorem 9.4, the improvement is that we do not need the boundary value to be piecewise smooth. The proof can be found in chapter 2 of [SS].

Finally let us convert Poisson's formula back to the rectangular coordinates as follows. Let $(x, y) = (r\cos\theta, r\sin\theta)$ be in D and $(x', y') = (\cos\alpha, \sin\alpha)$ a point on the boundary circle. The angle between these two vectors is given by $\theta - \alpha$ or $\alpha - \theta$. By the law of cosine,

$$\begin{aligned}(x - x')^2 + (y - y')^2 &= ||(x, y) - (x', y')||^2 \\ &= ||(x, y)||^2 + ||(x', y')||^2 - 2||(x, y)||||(x', y')||\cos(\theta - \alpha) \\ &= 1 + r^2 - 2r\cos(\theta - \alpha).\end{aligned}$$

On the other hand, the map $\alpha \mapsto (\cos\alpha, \sin\alpha), \alpha \in [-\pi, \pi]$, is a parametrisation of the unit circle. We have

$$ds = \sqrt{(\cos\alpha)'^2 + (\sin\alpha)'^2}\, d\alpha = d\alpha.$$

Therefore, Poisson's formula in rectangular coordinates is given by

$$u(x, y) = \frac{1 - r^2}{2\pi} \int_C \frac{\phi(x', y')}{(x - x')^2 + (y - y')^2}\, ds(x', y'), \quad (x, y) \in D,$$

where the integral is the line integral along the circle $C:\ r = 1$.